

# Legendre-Fenchel transforms in a nutshell

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The aim of this report is to list and explain the basic properties of the Legendre-Fenchel transform, which is a generalization of the Legendre transform commonly encountered in physics. The precise way in which the Legendre-Fenchel transform generalizes the Legendre transform is carefully explained and illustrated with many examples and pictures. The understanding of the difference between the two transforms is important because the general transform which arises in statistical mechanics is the Legendre-Fenchel transform, not the Legendre transform.

All the results contained in this report can be found with much more mathematical details and rigor in [1]. The proofs of these results can also be found in that reference.

## 1. Definitions

Consider a function  $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ . We define the **Legendre-Fenchel** (LF) transform of  $f(x)$  by the variational formula

$$f^*(k) = \sup_{x \in \mathbb{R}} \{kx - f(x)\}. \quad (1)$$

We also express this transform by  $f^*(k) = (f(x))^*$  or, more compactly, by  $f^* = (f)^*$ , where the star stands for the LF transform.

The LF transform of  $f^*(k)$  is

$$f^{**}(x) = \sup_{k \in \mathbb{R}} \{kx - f^*(k)\}. \quad (2)$$

This corresponds also to the **double LF transform** of  $f(x)$ . The double-star notation comes obviously from our compact notation for the LF transform:

$$f^{**} = (f^*)^* = ((f)^*)^*. \quad (3)$$

21 **Remark 1.** LF transforms can also be defined using an infimum (min) rather than a  
 22 supremum (max):

$$g^*(k) = \inf_{x \in \mathbb{R}} \{kx - g(x)\}. \quad (4)$$

23 Transforming one version of the LF transform to the other is just a matter of introducing  
 24 minus signs at the right place. Indeed,

$$-f^*(k) = -\sup_x \{kx - f(x)\} = \inf_x \{-kx + f(x)\}, \quad (5)$$

25 so that

$$g^*(q) = \inf_x \{qx - g(x)\}, \quad (6)$$

26 making the transformations  $g(x) = -f(x)$  and  $g^*(q) = -f^*(k = -q)$ .

27 **Remark 2.** The Legendre-Fenchel transform is often referred to in physics as the **Leg-**  
 28 **endre transform**. This does not do justice to Fenchel who explicitly studied the vari-  
 29 ational formula (1), and applied it to nondifferentiable as well as nonconvex functions.  
 30 What Legendre actually considered is the transform defined by

$$f^*(k) = kx_k - f(x_k) \quad (7)$$

31 where  $x_k$  is determined by solving

$$f'(x) = k. \quad (8)$$

32 This form is more limited in scope than the LF transform, as it applies only to differen-  
 33 tiable functions, and, we shall see later, convex functions. In this sense, the LF transform  
 34 is a generalization of the Legendre transform, which reduces to the Legendre transform  
 35 when applied to convex, differentiable functions. We shall comment more on this later.

36 **Remark 3.** The LF transform is not necessarily **self-inverse** (we also say **involutive**);  
 37 that is to say,  $f^{**}$  need not necessarily be equal to  $f$ . The equality  $f^{**} = f$  is taken for  
 38 granted too often in physics; we'll see later in which cases it actually holds and which  
 39 other cases it does not.

40 **Remark 4.** The definition of the LF transform can trivially be generalized to functions  
 41 defined on higher-dimensional spaces (i.e., functions  $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $n$  a positive  
 42 integer) by replacing the normal real-number product  $kx$  by the scalar product  $\mathbf{k} \cdot \mathbf{x}$ ,  
 43 where  $\mathbf{k}$  is a vector having the same dimension as  $\mathbf{x}$ .

44 **Remark 5.** (Steepest-descent or Laplace approximation). Consider the definite integral

$$F(k, n) = \int_{\mathbb{R}} e^{n[kx - f(x)]} dx. \quad (9)$$

45 In the limit  $n \rightarrow \infty$ , it is possible to approximate this integral using Laplace Method  
 46 (or steepest-descent method if  $x \in \mathbb{C}$ ) by locating the maximum value of the integrand  
 47 corresponding to the maximum value of the exponent  $kx - f(x)$  (assume there's only  
 48 one such value). This yields,

$$F(k, n) \approx \exp \left( n \sup_{x \in \mathbb{R}} \{kx - f(x)\} \right). \quad (10)$$

49 It can be proved that the corrections to this approximation are subexponential in  $n$ , i.e.,

$$\ln F(k, n) = n \sup_{x \in \mathbb{R}} \{kx - f(x)\} + o(n). \quad (11)$$

50 **Remark 6.** (The LF transform in statistical mechanics). Let  $U$  be the energy function  
 51 of an  $n$ -body system. In general, the density  $\Omega_n(u)$  of microscopic states of the system  
 52 having a mean energy  $u = U/n$  scales exponentially with  $n$ , which is to say that

$$\ln \Omega_n = ns(u) + o(n), \quad (12)$$

53 where  $s(u)$  is the microcanonical entropy function of the system. (This can be taken as  
 54 a definition of the microcanonical entropy.) Defining the canonical partition function of  
 55 the system in the usual way, i.e.,

$$Z_n(\beta) = \int \Omega_n(u) e^{-n\beta u} du, \quad (13)$$

56 we can use Laplace Method to write

$$\varphi(\beta) = \lim_{n \rightarrow \infty} -\frac{1}{n} \ln Z_n(\beta) = \inf_u \{\beta u - s(u)\}. \quad (14)$$

57 Physically,  $\varphi(\beta)$  represents the free energy of the system in the canonical ensemble. So,  
 58 what the above result shows is that the canonical free energy is the LF transform of the  
 59 microcanonical entropy ( $\varphi = s^*$ ). The inverse result, namely  $s = \varphi^*$ , is not always true,  
 60 as will become clear later.

## 61 2. Theory of LF transforms

62 The theory of LF transforms deals mainly with two questions:

63 Q1: How is the shape of  $f^*(k)$  determined by the shape of  $f(x)$ , and vice versa?

64 Q2: When is the LF transform involutive? That is, when does  $f^{**} = ((f)^*)^* = f$ ?

65 We'll see next that these two questions are answered by using a fundamental concept of  
66 convex analysis known as a supporting line.

### 67 2.1. Supporting lines

68 We say that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has or admits a **supporting line** at  $x \in \mathbb{R}$  if there  
69 exists  $\alpha \in \mathbb{R}$  such that

$$f(y) \geq f(x) + \alpha(y - x), \quad (15)$$

70 for all  $y \in \mathbb{R}$ . The parameter  $\alpha$  is the slope of the supporting line. We further say that a  
71 supporting line is **strictly supporting** at  $x$  if

$$f(y) > f(x) + \alpha(y - x) \quad (16)$$

72 holds for all  $y \neq x$ . For these definitions to make sense, we need obviously to have  
73  $f < \infty$ .

74 **Remark 7.** For convenience, it is useful to replace the expression “ $f$  admits a support-  
75 ing line at  $x$ ” by “ $f$  is **convex** at  $x$ ”. So, from now on, the two expressions mean the  
76 same (this is a definition). If  $f$  does not admit a supporting line at  $x$ , then we shall say  
77 that  $f$  is **nonconvex** at  $x$ .

78 The geometrical interpretation of supporting lines is shown in Figure 1. In this figure,  
79 we see that

- 80 • The point  $a$  admits a supporting line ( $f$  is convex at  $a$ ). The supporting line has  
81 the property that it touches  $f$  at the point  $(a, f(a))$  and lies *beneath* the graph of  
82  $f(x)$  for all  $x$ ; hence the term “supporting”.
- 83 • The supporting line at  $a$  is strictly supporting because it touches the graph of  $f(x)$   
84 only at  $a$ . In this case, we say that  $f$  is strictly convex at  $a$ .
- 85 • The point  $b$  does not admit any supporting lines; any lines passing through  $(b, f(b))$   
86 must cross the graph of  $f(x)$  at some point. In this case, we also say that  $f$  is non-  
87 convex at  $b$ .

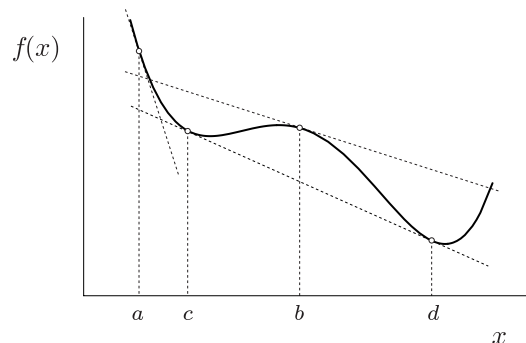


Figure 1: Geometric interpretation of supporting lines.

- 88 • The point  $c$  admits a supporting line which is non-strictly supporting, as it touches  
 89 another point ( $d$ ) of the graph of  $f(x)$ . (The points  $c$  and  $d$  share the same sup-  
 90 porting line.)

91 From this picture, we easily deduce the following result:

92 **Proposition 1.** If  $f$  admits a supporting line at  $x$  and  $f'(x)$  exists, then the slope  $\alpha$  of  
 93 the supporting line must be equal to  $f'(x)$ . In other words, for differentiable functions,  
 94 a supporting line is also a **tangent** line.

## 95 2.2. Convexity properties

96 Before answering Q1 and Q2, let us pause briefly for two important results, which we  
 97 state without proofs.

98 **Theorem 2.**  $f^*(k)$  is an always convex function of  $k$  (independently of the shape of  $f$ ).

99 **Corollary 3.**  $f^{**}(x)$  is an always convex function of  $x$  (again, independently of the  
 100 shape of  $f$ ).

101 The precise meaning of convex here is that  $f^*$  (or  $f^{**}$ ) admits a supporting line at  
 102 all  $k$  (all  $x$ , respectively). More simply, it means that  $f^*$  and  $f^{**}$  are  $\cup$ -shaped.<sup>1</sup>

103 Note that these results tell us already that the LF transform cannot always be involu-  
 104 tive. Indeed,  $f^{**}(x)$  is convex even if  $f(x)$  is not, so that  $f \neq f^{**}$  if  $f$  is not everywhere  
 105 convex. We'll see later that this is the only problematic case.

<sup>1</sup>There seems to be some confusion in the literature about the definitions of “concave” and “convex.” The Webster (7th Edition), for one, defines a  $\cup$ -shaped function to be *concave* rather than convex. However, most mathematical textbooks will agree in defining the same function to be convex.

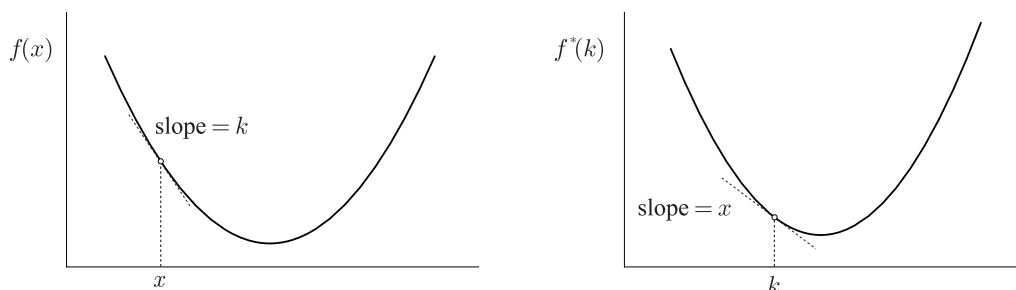


Figure 2: Illustration of the duality property for supporting lines: points of  $f$  are transformed into slopes of  $f^*$ , and slopes of  $f$  are transformed into points of  $f^*$ .

### 106 2.3. Supporting line duality

107 We now answer our first question (Q1): How is the shape of  $f^*(k)$  determined by the  
 108 shape of  $f(x)$ , and vice versa? A partial answer is provided by the following result:

109 **Theorem 4.** If  $f$  admits a supporting line at  $x$  with slope  $k$ , then  $f^*$  at  $k$  admits a  
 110 supporting line with slope  $x$ .

111 This theorem is illustrated in Figure 2. The next theorem covers the special case of  
 112 strict convexity.

113 **Theorem 5.** If  $f$  admits a strict supporting line at  $x$  with slope  $k$ , then  $f^*$  admits a  
 114 tangent supporting line at  $k$  with slope  $f^{*'}(k) = x$ . (Hence  $f^*$  is differentiable in this  
 115 case in addition to admit a supporting line.)

### 116 2.4. Inversion of LF transforms

117 The answer to Q2 ( $f \stackrel{?}{=} f^{**}$ ) is provided by the following result:

118 **Theorem 6.**  $f(x) = f^{**}(x)$  if and only if  $f$  admits a supporting line at  $x$ .

119 Thus, from the point of view of  $f(x)$ , we have that the LF transform is involutive at  
 120  $x$  if and only if  $f$  is convex at  $x$  (in the sense of supporting lines). Changing our point of  
 121 view to  $f^*(k)$ , we have the following:

122 **Theorem 7.** If  $f^*$  is differentiable at  $k$ , then  $f = f^{**}$  at  $x = f^{*'}(k)$ .

123 We'll see later with a specific example that the differentiability property of  $f^*$  is  
 124 sufficient (as stated) but non-necessary for  $f = f^{**}$ . For now, we note the following  
 125 obvious corollary:

126 **Corollary 8.** If  $f^*(k)$  is everywhere differentiable, then  $f(x) = f^{**}(x)$  for all  $x$ .

127 This says in words that the LF transform is completely involutive if  $f^*(k)$  is every-  
128 where differentiable.

129 We end this section with another corollary and a result which helps us visualize the  
130 meaning of  $f^{**}(x)$ .

131 **Corollary 9.** A convex function can always be written as the LF transform of another  
132 function. (This is not true for nonconvex functions.)

133 **Theorem 10.**  $f^{**}(x)$  is the largest convex function satisfying  $f^{**}(x) \leq f(x)$ .

134 Because of this result, we call  $f^{**}(x)$  the **convex envelope** or **convex hull** of  $f(x)$ .  
135 We'll precise the meaning of these expressions in the next section.

### 136 3. Some particular cases

137 We consider in this section a number of examples which help visualize the meaning  
138 and application of all the results presented in the previous section. All of the examples  
139 considered arise in statistical mechanics.

#### 140 3.1. Differentiable, convex functions

141 The LF transform

$$f^*(k) = \sup_x \{kx - f(x)\} \quad (17)$$

142 is in general evaluated by finding the critical points  $x_k$  (there could be more than one)  
143 which maximize the function

$$F(x, k) = kx - f(x). \quad (18)$$

144 In mathematical notation, we express  $x_k$  in the following manner:

$$x_k = \arg \sup_x F(x, k) = \arg \sup_x \{kx - f(x)\}, \quad (19)$$

145 where “arg sup” reads “arguments of the supremum,” and mean in words “points at which  
146 the maximum occurs.”

147 Now, assume that  $f(x)$  is everywhere differentiable. Then, we can find the maximum  
148 of  $F(x, k)$  using the common rules of calculus by solving

$$\frac{\partial}{\partial x} F(x, k) = 0, \quad (20)$$

149 for a fixed value of  $k$ . Given the form of  $F(x, k)$ , this is equivalent to solving

$$k = f'(x) \tag{21}$$

150 for  $x$  given  $k$ . As noted before, there could be more than one critical points of  $F(x, k)$   
 151 that would solve here the above differential equation. To make sure that there is actually  
 152 only one solution for every  $k \in \mathbb{R}$ , we need to impose the following two conditions on  
 153  $f$ :

- 154 1.  $f'(x)$  is continuous and monotonically increasing for increasing  $x$ ;
- 155 2.  $f'(x) \rightarrow \infty$  for  $x \rightarrow \infty$  and  $f'(x) \rightarrow -\infty$  for  $x \rightarrow -\infty$ .

156 Given these, we are assured that there exists a unique value  $x_k$  for each  $k \in \mathbb{R}$  satisfying  
 157  $k = f'(x_k)$  and which maximizes  $F(x, k)$ . As a result, we can write

$$f^*(k) = kx_k - f(x_k), \tag{22}$$

158 where

$$f'(x_k) = k. \tag{23}$$

159 These two equations define precisely what the Legendre transform of  $f(x)$  is (as  
 160 opposed to the LF transform, which is defined with the sup formula). Accordingly, we  
 161 have proved that the LF transform reduces to the Legendre transform for differentiable  
 162 and strictly convex functions. (The strictly convex property results from the monoton-  
 163 ity of  $f'(x)$ .) Since  $f(x)$  at this point is convex by assumption, we must have  $f = f^{**}$   
 164 for all  $x$ . Therefore, the Legendre transform must be involutive (always), and the inverse  
 165 Legendre transform is the Legendre transform itself; in symbol,

$$f(x) = k_x x - f^*(k_x), \tag{24}$$

166 where  $k_x$  is the unique solution of

$$f^{*'}(k) = x. \tag{25}$$

### 167 3.2. Function having a nondifferentiable point

168 What happens if  $f(x)$  has one or more nondifferentiable points? Figure 3 shows a partic-  
 169 ular example of a function  $f(x)$  which is nondifferentiable at  $x_c$ . What does its LF  
 170 transform  $f^*(k)$  look like?

171 The answer is provided by what we have learned about supporting lines. Let's con-  
 172 sider the differentiable and nondifferentiable parts of  $f(x)$  separately:



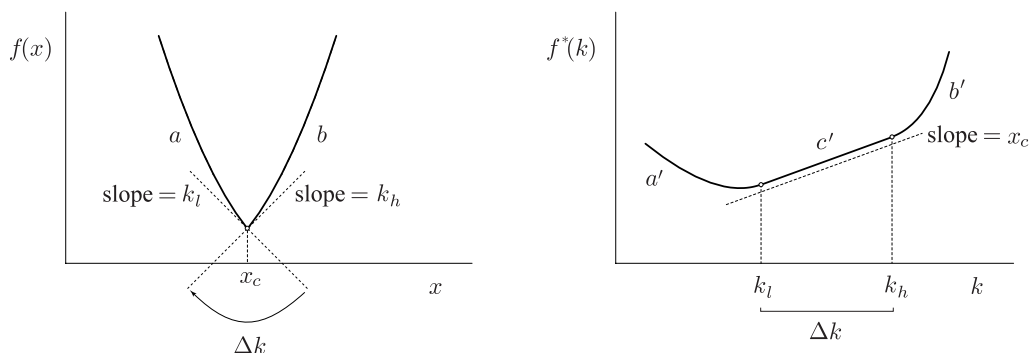


Figure 3: Function having a nondifferentiable point; its LF transform is affine.

- 173 • Differentiable points of  $f$ : Each point  $(x, f(x))$  on the differentiable branches of  
 174  $f(x)$  admits a strict supporting line with slope  $f'(x) = k$ . From the results of  
 175 the previous section, we then know that these points are transformed at the level  
 176 of  $f^*(k)$  into points  $(k, f^*(k))$  admitting supporting line of slopes  $f^{*'}(k) = x$ .  
 177 For example, the differentiable branch of  $f(x)$  on the left (branch  $a$  in Figure  
 178 3) is transformed into a differentiable branch of  $f^*(k)$  (branch  $a'$ ) which extends  
 179 over all  $k \in (-\infty, k_l]$ . This range of  $k$ -values arises because the slopes of the left-  
 180 branch of  $f(x)$  ranges from  $-\infty$  to  $k_l$ . Similarly, the differentiable branch of  $f(x)$   
 181 on the right (branch  $b$ ) is transformed into the right differentiable branch of  $f^*(k)$   
 182 (branch  $b'$ ), which extends from  $k_h$  to  $+\infty$ . (Note that, for the two differentiable  
 183 branches, the LF transform reduces to the Legendre transform.)
- 184 • Nondifferentiable point of  $f$ : The nondifferentiable point  $x_c$  admits not one but  
 185 infinitely many supporting lines with slopes in the range  $[k_l, k_h]$ . As a result, each  
 186 point of  $f^*(k)$  with  $k \in [k_l, k_h]$  must admit a supporting line with constant slope  
 187  $x_c$  (branch  $c'$ ). That is,  $f^*(k)$  must have a constant slope  $f^{*'}(k) = x_c$  in the  
 188 interval  $[k_l, k_h]$ . We say in this case that  $f^*(k)$  is **affine** or **linear** over  $(k_l, k_h)$ .  
 189 (The affinity interval is always the open version of the interval over which  $f^*$  has  
 190 constant slope.)

191 The case of functions having more than one nondifferentiable point is treated simi-  
 192 larly by considering each nondifferentiable point separately.

### 193 3.3. Affine function

194 Since  $f(x)$  in the previous example is convex,  $f(x) = f^{**}(x)$  for all  $x$ , and so the roles  
 195 of  $f$  and  $f^*$  can be inverted to obtain the following: a convex function  $f(x)$  having an

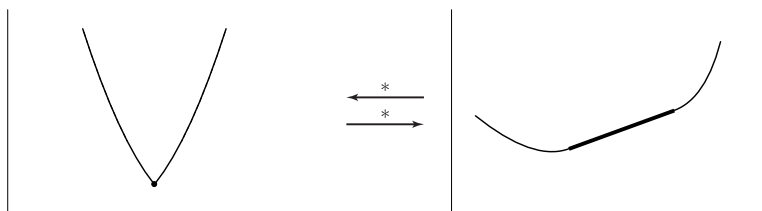


Figure 4: Nondifferentiable points are transformed into affine parts under the action of the LF transform and vice versa.

196 affine part has a LF transform  $f^*(k)$  having one nondifferentiable point; see Figure 4.  
 197 More precisely, if  $f(x)$  is affine over  $(x_l, x_h)$  with slope  $k_c$  in that interval, then  $f^*(k)$   
 198 will have a nondifferentiable point at  $k_c$  with left- and right-derivatives at  $k_c$  given by  $x_l$   
 199 and  $x_h$ , respectively.

### 200 3.4. Bounded-domain function with infinite slopes at boundaries

201 Consider the function  $f(x)$  shown in Figure 5. This function has the particularity to be  
 202 defined only on a bounded domain of  $x$ -values, which we denote by  $[x_l, x_h]$ . Further-  
 203 more,  $f'(x) \rightarrow \infty$  as  $x \rightarrow x_l + 0$  and  $x \rightarrow x_h - 0$  (the derivative of  $f$  blows up near at  
 204 the boundaries). Outside the interval of definition of  $f(x)$ , we formally set  $f(x) = \infty$ .

205 To determine the shape of  $f^*(k)$ , we use again what we know about supporting lines  
 206 of  $f$  and  $f^*$ . All points  $(x, f(x))$  with  $x \in (x_l, x_h)$  admit a strict supporting line with  
 207 slope  $k(x)$ . These points are represented at the level of  $f^*$  by points  $(k(x), f^*(k(x)))$   
 208 having a supporting line of slope  $x$ . As  $x$  approaches  $x_l$  from the right, the slope of  $f(x)$   
 209 diverges to  $-\infty$ . At the level of  $f^*$ , this implies that the slope of the supporting line of  
 210  $f^*$  reaches  $x_l$  as  $k \rightarrow -\infty$ . Similarly, since the slope of  $f(x)$  goes to  $+\infty$  as  $x \rightarrow x_h$ ,  
 211 the slope of the supporting line of  $f^*$  reaches the value  $x_h$  as  $k \rightarrow +\infty$ ; see Figure 5.

212 Note, finally, that  $f = f^{**}$  since  $f$  is convex. This means that we can invert the roles  
 213 of  $f$  and  $f^*$  in this example just like in the previous one to obtain the following: the LF  
 214 transform of a convex function which is asymptotically linear is a convex function which  
 215 is finite on a bounded domain with diverging slopes at the boundaries.

### 216 3.5. Bounded-domain function with finite slopes at boundaries

217 Consider now a variation of the previous example. Rather than having diverging slopes  
 218 at the boundaries  $x_l$  and  $x_h$ , we assume that  $f(x)$  has finite slopes at these points. We  
 219 denote the right-derivative of  $f$  at  $x_l$  by  $k_l$  and its left-derivative at  $x_h$  by  $k_h$ .

220 For this example, everything works as in the previous example except that we have  
 221 to be careful about the boundary points. As in the case of nondifferentiable points,  $f$  at

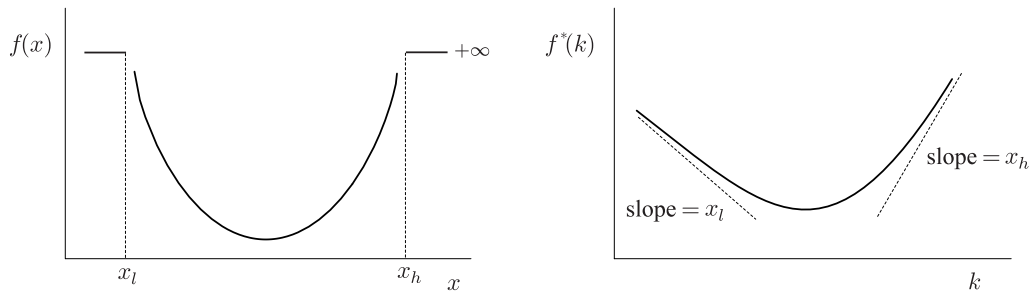


Figure 5: Function defined on a bounded domain with diverging slopes at boundaries; its LF transform is asymptotically linear as  $|k| \rightarrow \pm\infty$ .

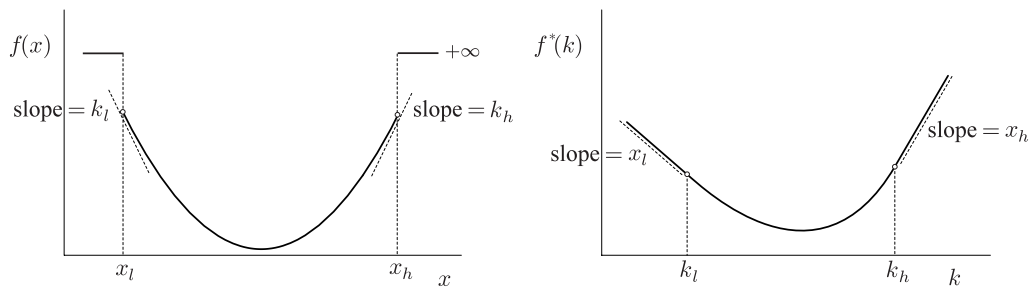


Figure 6: Function defined on a bounded domain with finite slopes at boundaries; its LF transform has affine parts outside some interior domain.

222  $x_h$  admits not one but infinitely many supporting lines with slopes taking values in the  
 223 range  $[k_h, \infty)$ . At the level of  $f^*$ , this means that all points  $(k, f^*(k))$  with  $k \in [k_h, \infty)$   
 224 have supporting lines with constant slope  $x_h$ ; that is,  $f^*(k)$  is affine past  $k_h$  with slope  
 225  $x_h$ . Likewise,  $f$  at  $x_l$  admits an infinite number of supporting lines with slopes now  
 226 ranging from  $-\infty$  to  $k_l$ . As a consequence,  $f^*$  must be affine over the range  $(-\infty, k_l)$   
 227 with constant slope  $x_l$ ; see Figure 6.

### 228 3.6. Nonconvex function

229 Our last example is quite interesting, as it illustrates the precise case for which the LF  
 230 transform is not involutive, namely nonconvex functions.

231 The function that we consider is shown in Figure 7; it has three branches having the  
 232 following properties:

- 233 • Branch  $a$ : The points on this branch, which extends from  $x = -\infty$  to  $x_l$ , admit  
 234 strict supporting lines. This branch is thus transformed into a differentiable branch

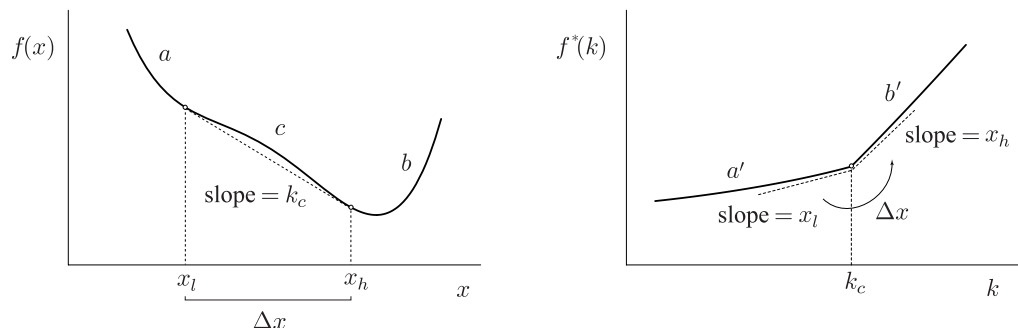


Figure 7: Nonconvex function; its LF transform has a nondifferentiable point.

235 at the level of  $f^*$  (branch  $a'$ ).

236 • Branch  $b$ : Similarly as for branch  $a$ .

237 • Branch  $c$ : None of the points on this branch, which extends from  $(x_l, x_h)$ , admit  
 238 supporting lines. This means that these points are not represented at the level of  
 239  $f^*$ . In other words, there is not one point of  $f^*$  which admits a supporting line  
 240 with slope in the range  $(x_l, x_h)$ . (That would contradict the fact that  $f^*$  has a  
 241 supporting line at  $k$  with slope  $x$  if and only if  $f$  admits a supporting line at  $x$  with  
 242 slope  $k$ .)

243 These three observations have two important consequences (see Figure 7):

- 244 1.  $f^*(k)$  must have a nondifferentiable point at  $k_c$ , with  $k_c$  equal to the slope of the  
 245 supporting line connecting the two points  $(x_l, f(x_l))$  and  $(x_h, f(x_h))$ . This follows  
 246 since  $x_l$  and  $x_h$  share the same supporting line of slope  $k_c$ . Thus, in a way,  $f^*$  must  
 247 have two slopes at  $k_c$ .
- 248 2. Define the **convex extrapolation** of  $f(x)$  to be the function obtained by replacing  
 249 the nonconvex branch of  $f(x)$  (branch  $c$ ) by the supporting line connecting the  
 250 two convex branches of  $f$  ( $a$  and  $b$ ). Then, both the LF transforms of  $f$  and  
 251 its convex extrapolation yield  $f^*$ . This is evident from our previous working of  
 252 nondifferentiable and affine functions. It should also be evident from the example  
 253 of nondifferentiable functions that the convex extrapolation of  $f$  is nothing but  
 254  $f^{**}$ , the double LF transform of  $f$ . This explains why we call  $f^{**}$  the convex  
 255 envelope of  $f$ .

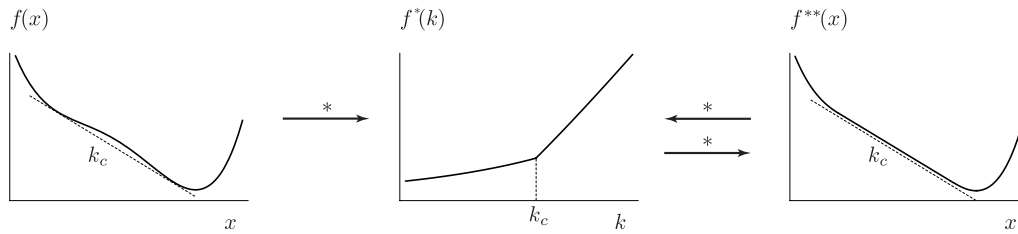


Figure 8: Structure of the LF transform for nonconvex functions.

256 To summarize, note that, as a result of Point 2 above, we have

$$(f^{**})^* = (f)^* = f^*. \quad (26)$$

257 Also, for the example considered, we have

$$(f^*)^* = f^{**} \neq f. \quad (27)$$

258 Overall, this means that the LF transform has the following structure:

$$f \rightarrow f^* \rightleftharpoons f^{**}, \quad (28)$$

259 where the arrows stand for the LF transform; see Figure 8. This diagram clearly shows  
 260 that the LF transform is non-involutive in general. For convex functions, i.e., functions  
 261 admitting supporting lines everywhere, the diagram reduces to

$$f \rightleftharpoons f^*. \quad (29)$$

262 That is, in this case, the LF transform is involutive (see Theorem 2.4).

#### 263 4. Important results to remember

- 264 • The LF transform yields only convex functions:  $f^* = (f)^*$  is convex and so is  
 265  $f^{**} = (f^*)^*$ .
- 266 • The shape of  $f^*$  is determined from the shape of  $f$  by using the duality relation-  
 267 ship which exists between the supporting lines of  $f^*$  and those of  $f$ .
  - 268 – Points of  $f$  are transformed into slopes of  $f^*$ , and slopes of  $f$  are trans-  
 269 formed into points of  $f^*$ .
  - 270 – Nondifferentiable points of  $f$  are transformed, through the action of the LF  
 271 transform, into affine branches of  $f^*$ .

272 – Affine or nonconvex branches of  $f$  are transformed into nondifferentiable  
 273 points of  $f^*$ . These are the only two cases producing nondifferentiable  
 274 points.

275 • The involution (self-inverse) property of the LF transform is determined from the  
 276 supporting line properties of  $f$  or from the differentiability properties of  $f^*$ .

277 –  $f = f^{**}$  at  $x$  if and only if  $f$  admits a supporting line at  $x$ .

278 – If  $f^*$  is differentiable at  $k$ , then  $f = f^{**}$  at  $x = f^*(k)$ .

279 • The double LF transform  $f^{**}$  of  $f$  corresponds to the convex envelope of  $f$ .

280 • The complete structure of the LF transform for general functions goes as follows:

$$f \rightarrow f^* \rightleftharpoons f^{**}, \quad (30)$$

281 where the arrows denote the LF transform. For convex functions ( $f = f^{**}$ ), this  
 282 reduces to

$$f \rightleftharpoons f^*; \quad (31)$$

283 i.e., in this case, the LF transform is involutive.

284 • The LF transform is more general than the Legendre transform because it applies  
 285 to nonconvex functions as well as nondifferentiable functions.

286 • The LF transform reduces to the Legendre transform in the case of convex, differ-  
 287 entiable functions.

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## 294 References

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