Chapter 7: Quantum Statistics
Part II: Applications

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1. Fermi Gas
   - Fermi-Dirac Distribution, Degenerate Fermi Gas
   - Electrons in Metals

2. Properties at $T = 0K$
   - Properties at $T = 0K$
   - Total energy of all electrons

3. Properties at $T \neq 0K$
   - Properties at $T \neq 0K$
   - Heat Capacity
Fermion systems

Physics systems that can be described by Fermion Gas model:

- $^3$He atoms
- proton or/and neutron in atomic nucleus
- electrons in white dwarf stars
- neutrons in neutron star
- electron gas in metals

We consider cases when $\frac{V}{N} \ll v_Q$, unlike in cases when the Boltzmann Statistics may apply when $\frac{V}{N} \gg v_Q$. For example, the quantum volume for an electron at room temperature (300K) is:

$$v_Q = \left( \frac{h}{\sqrt{2\pi mkT}} \right)^3 \approx (4.3nm)^3$$

In metal, the volume per conduction electron is about the volume of an atom, about $(0.2nm)^3$
Fermion systems - cnt.

Let’s take a look at the Fermi-Dirac distribution again.

\[ \bar{n}_{FD} = \frac{1}{e^{(\epsilon - \mu)/kT} + 1} \]  

(1)

Note-1: \( T \rightarrow 0 \), the FD distribution becomes a step function.

Note-2: The macrostate of the system is **completely defined** if we know the mean **occupancy number** for all energy levels: \( f(E) = \bar{n}(E) \).

Note-3: Fermi energy \( \epsilon_F \): the highest energy of all electrons. **Occupancy** = 0 when \( \epsilon > \epsilon_F \)
Electrons in Metals

- Energy to remove one electron from the metal is about a few eV.
- Electrons on "out-layer" states: they move freely in solid metal. They have density of about 1/perm ion.
- In the conductive band, the density of electrons is about $10^{29}/m^3$.
- Can we treat these electrons as Fermion Gas? Two objections:
  - Objection (1) The Coulomb interactions between electrons at this density must be extremely strong.
  - Objection (2) In a solid, the electrons move in the strong electric fields of the positive ions.
The answer is yes:

**Landau’s Fermi liquid theory:** It is addressed by the Landau Fermi liquid theory.

**Altered density of electron states:** It is that while the field of ions alters the density of states and the effective mass of the electrons, it does not otherwise affect the validity of the ideal gas approximation. Thus, in the case of simple metals, it is safe to consider the mobile charge carriers as electrons with the mass slightly renormalized by interactions. There are, however, examples that the interactions lead to the mass enhancement by a factor of 100-1000 (heavy fermions).
Properties at $T = 0K$

FD distribution is a simple step function at $T = 0$. Let’s start from this simple situation.

1-d potential well: $\lambda_n = \frac{2L}{n}$, $p_n = \frac{h}{\lambda_n} = \frac{hn}{2L}$

3-d Box: $p_x = \frac{hn_x}{2L_x}$, $p_y = \frac{hn_y}{2L_y}$, $p_z = \frac{hn_z}{2L_z}$, $n_x, n_y, n_z = 0, 1, 2, \ldots$

Allowed energy: $\epsilon = \frac{|\vec{p}|^2}{2m} = \frac{h^2}{8ml^2}(n_x^2 + n_y^2 + n_z^2)$

Each vector in n-space corresponds to a set of $\{n_x, n_y, n_z\}$
Chemical potential, Fermi Energy

\[ \mu = \epsilon_F = \frac{\hbar^2}{8mL^2} n_{max}^2, \]
where \( n_{max}^2 = (n_x^2 + n_y^2 + n_z^2)_{max}. \)

The total number of occupied states are:

\[ N = 2 \times \text{(Volume of the } \frac{1}{8} \text{-sphere}) = 2 \cdot \frac{1}{8} \cdot \frac{4}{3} \pi n_{max}^3 = \frac{\pi n_{max}^3}{3}. \]

The first 2 is the due to the fact that electron has two spin states. So,

\[ \epsilon_F = \frac{\hbar^2}{8mL^2} \left( \frac{3N}{\pi} \right)^{2/3} = \frac{h^2}{8m} \left( \frac{3N}{\pi V} \right)^{2/3} \tag{2} \]

This is the highest energy of all electrons: states above this energy have zero occupancy.

How big is it? \( \epsilon_F = \frac{\hbar^2}{8m} \left( \frac{3}{\pi} n \right)^{2/3} = \frac{(6.6 \times 10^{-34})^2}{8.9 \times 10^{-31}} \cdot (10^{29})^{2/3} = 10^{-18} \text{ J} \approx 6 \text{ eV}. \)

Fermi Temperature: \( T_F = \frac{\epsilon_F}{k} = \frac{6 \text{ eV}}{8.6 \times 10^{-5} \text{ eV/K}} = 10^4 \sim 10^5 K. \)
We have obtained $\epsilon = \frac{h^2}{8mL^2} n^2$, Eq. (7.36) in the textbook, p.273. From this, we get the total energy of all electrons:

$$U = 2 \sum_{n_x, n_y, n_z} \epsilon(\bar{n}) \approx 2 \int \int \int \epsilon(n_x, n_y, n_z) dn_x dn_y dn_z$$

(3)

2: electron two spin states.

$$U \approx 2 \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin \theta d\theta \int_0^{n_{\text{max}}} \epsilon(n) n^2 dn$$

(4)

$$U \approx \pi \int_0^{n_{\text{max}}} \epsilon(n) n^2 dn = \frac{\pi h^2}{8mL^2} \int_0^{n_{\text{max}}} n^4 dn$$

(5)

$$U \approx \frac{\pi h^2 n_{\text{max}}^5}{40mL^2} = \frac{3}{5} N\epsilon_F$$

(6)

Here we used Eq. (7.38, 7.39): $N = \frac{\pi n_{\text{max}}^3}{3}$ and $\epsilon_F = \frac{h^2}{8m} \left( \frac{3N}{\pi V} \right)^{2/3}$. 
 Degeneracy Pressure

Ideal Gas: \( PV = kNT \). When \( T \to 0 \), \( P \to 0 \).
We have obtained the total energy of all electrons in the system \( U \), which can be re-written as

\[
U = \frac{\pi h^2}{40mL^2} \cdot \left( \frac{3N}{\pi} \right)^{5/3}
\]

(7)

\[
U = \frac{3}{5} N \frac{h^2}{8m} \cdot \left( \frac{3N}{\pi} \right)^{2/3} \cdot \left( \frac{1}{V} \right)^{2/3}
\]

(8)

\[
(9)
\]

Since pressure \( P = -\frac{\partial U}{\partial V} \), so

\[
P = \frac{3}{5} N \frac{h^2}{8m} \cdot \left( \frac{3N}{\pi} \right)^{2/3} \cdot \left( \frac{1}{V} \right)^{5/3} 2 \frac{2}{3}
\]

(10)

\[
P = 2 \frac{N\epsilon_F}{5V} = 2\frac{U}{3V}
\]

(11)
Degeneracy Pressure - cnt.

This $P$ is called "Degeneracy Pressure": $P = \frac{2}{5} \frac{N \epsilon_F}{V} = \frac{2U}{3V}$

- It is a positive pressure: supporting the structure of matter, prevent matter from collapsing due to the electric attracting force between electrons and ions.
- It has nothing to do with e-e repulsion
- It is very big: for typical metal,
  \[ P = \frac{2}{5} n \epsilon_F \approx 10^{29} \text{ m}^{-3} \cdot 5 \cdot 10^{-19} \text{ J} = 5 \times 10^{10} \text{ Pa}. \]
- In experiment, we can measure the "Bulk modulus",
  \[ B = -V \left( \frac{\partial P}{\partial V} \right)_T. \]

We can prove $B = \frac{10}{9} \frac{U}{V}$.
Let’s take this as an in-class exercise.
Degeneracy Pressure - cnt.

\[ P = \frac{2}{5} N \frac{h^2}{8m} \left( \frac{3N}{\pi} \right)^{2/3} \left( \frac{1}{V} \right)^{5/3} \]  
(12)

\[ \left( \frac{\partial P}{\partial V} \right)_T = \frac{2}{5} \left( \frac{5}{3} \right) N \frac{h^2}{8m} \left( \frac{3N}{\pi} \right)^{2/3} \left( \frac{1}{V} \right)^{8/3} \]  
(13)

\[ \left( \frac{\partial P}{\partial V} \right)_T = -\frac{2}{3} N \frac{h^2}{8m} \left( \frac{3N}{\pi} \right)^{2/3} \frac{1}{V^{2/3}} \frac{1}{V^2} \]  
(14)

\[ \left( \frac{\partial P}{\partial V} \right)_T = -\frac{2}{3} N\epsilon_F \frac{1}{V^2} = -\frac{2}{3} \frac{5}{3} U \frac{1}{V^2} = -\frac{10}{9} \frac{U}{V^2} \]  
(15)

Therefore,

\[ B = -V \left( \frac{\partial P}{\partial V} \right)_T = \frac{10}{9} \frac{U}{V}. \]  
(16)
Degeneracy Pressure - cnt.

What we have learned is the non-relativistic case:
\[ P = \frac{2}{3} \frac{U}{V} = \frac{2}{5} \frac{N}{V} \epsilon_F. \]
Using \( \epsilon_F = \frac{\hbar^2}{8m} \left( \frac{3N}{\pi V} \right)^{2/3} \), \( P = \frac{1}{20} \frac{\hbar^2}{m} \left( \frac{3}{\pi} \right)^{2/3} n^{5/3} \propto n^{5/3}. \)

This is non-relativistic case. In a white dwarf or a neutron star, this relation breaks down.

The relativistic approximation is \( P \propto n^{4/3}. \)

\[
\begin{align*}
\text{Log } (T / \text{K}) & \quad \text{Radiation pressure} \\
\text{Log } (\rho / \text{gcm}^{-3}) \\
\text{Degenerate, relativistic} & \quad \text{Degenerate, non-relativistic} \\
\end{align*}
\]

\[ P_{\text{relativistic}} = P_{\text{non-relativistic}} \text{ happens at } n_e = 10^{36} \text{ m}^{-3}. \]
### Mass of stars

**Table:** Star mass and its destiny

<table>
<thead>
<tr>
<th>Mass Range</th>
<th>White Dwarf</th>
<th>Neutron Star</th>
<th>Black Hole</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M &lt; 1.4M_\odot$</td>
<td>white dwarf</td>
<td>neutron star</td>
<td>black hole</td>
</tr>
<tr>
<td>$1.4M_\odot &lt; M &lt; 3M_\odot$</td>
<td>e-degeneracy pressure</td>
<td>neutron degeneracy pressure</td>
<td>gravity wins</td>
</tr>
<tr>
<td>$M &gt; 3M_\odot$</td>
<td>like metal</td>
<td>nuclear matter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>the cause: $e^- + p \rightarrow n$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Density of States

We have obtained this expressions of total energy: \( U \approx \pi \int_0^{n_{\text{max}}} \epsilon(n)n^2 \, dn \)

Since \( \epsilon = \frac{h^2}{8mL^2} n^2 \)

\[ n^2 = \frac{8mL^2}{h^2} \epsilon, \quad n = \sqrt{\frac{8mL^2}{h^2} \epsilon}, \]

\[ dn = \sqrt{\frac{8mL^2}{h^2} \frac{1}{2\sqrt{\epsilon}}} \, d\epsilon \]

The total energy \( U \) can be calculated by integrating over \( \epsilon \):

\[ U \approx \pi \int_0^{\epsilon_F} \epsilon \cdot \frac{8mL^2}{h^2} \epsilon \cdot \sqrt{\frac{8mL^2}{h^2} \frac{1}{2\sqrt{\epsilon}}} \, d\epsilon \]

\[ U \approx \int_0^{\epsilon_F} \epsilon \cdot \left[ \frac{\pi}{2} \left( \frac{8mL^2}{h^2} \right)^{3/2} \sqrt{\epsilon} \right] \, d\epsilon \]

\[ U \approx \int_0^{\epsilon_F} \epsilon \cdot g(\epsilon) \, d\epsilon \]

in which \( g(\epsilon) = \frac{\pi}{2} \left( \frac{8mL^2}{h^2} \right)^{3/2} \sqrt{\epsilon} \). Or,

\[ g(\epsilon) = \frac{\pi (8m)^{3/2}}{2h^3} V \sqrt{\epsilon} = \frac{3N}{2\epsilon_F^{3/2}} \sqrt{\epsilon} \quad (17) \]
Density of States - cnt.

Remarks:
(1) $g(\epsilon)$ is called the "density of states": The number of single-particle states per unit energy interval.

(2) It gives $\int_0^{\epsilon_F} g(\epsilon) d\epsilon = N$, as expected.

(3) The curve of $g(\epsilon)$.

\begin{center}
\includegraphics{density_of_states_graph.png}
\end{center}
Exercise 07-03
Consider a degenerate electron gas in which all electrons are highly relativistic, i.e. $\epsilon \gg mc^2$, so that their energy is $\epsilon = pc$ where $p$ is the magnitude of the momentum vector.
(a) Show that for a relativistic electron gas at zero temperature, the chemical potential (or the Fermi energy) is given by $\mu = \hbar c \left( \frac{3N}{8\pi V} \right)^{1/3}$.
(b) Find a formula for the total energy of this system in terms of $N$, and $\mu$. 
Exercise 07-04: Electron Degenerate Pressure and White Dwarf Star
(Problem 7.23)

By combining the ideas of relativity and quantum mechanics, Chandrasekhar made important contributions to our understanding of the star evolution.

The Fermi pressure of a degenerate electron gas prevents the gravitational collapse of the star if the star is not too massive (the white dwarf).

White Dwarf
Electrons run out of room to move around. Electrons prevent further collapse. Protons & neutrons still free to move around.

Stronger gravity => more compact.

Neutron Star
Electrons + protons combine to form neutrons. Neutrons run out of room to move around. Neutrons prevent further collapse. Much smaller!

Black Hole
Gravity wins! Nothing prevents collapse.
Properties at $T \neq 0K$

When $T \neq 0$ $k$, the F.D. distribution is no longer a step function. See figure on the right.
In stead of $N = \int_{0}^{\epsilon_F} g(\epsilon) d\epsilon$, the generic expression is:

\[
N = \int_{0}^{\infty} g(\epsilon) \tilde{n}_{FD}(\epsilon) d\epsilon = \int_{0}^{\infty} g(\epsilon) \frac{1}{e^{(\epsilon-\mu)/kT} + 1} d\epsilon \quad (18)
\]

\[
U = \int_{0}^{\infty} \epsilon g(\epsilon) \tilde{n}_{FD}(\epsilon) d\epsilon = \int_{0}^{\infty} \epsilon g(\epsilon) \frac{1}{e^{(\epsilon-\mu)/kT} + 1} d\epsilon \quad (19)
\]

How to do the integrations?
- A method first proposed by Arnold Sommerfeld, called the "Sommerfeld Expansion".
We will confine our attention to the limit in which $\mu$ is close to its $T=0$ value, the Fermi energy $\varepsilon_F$. 

Diagram:

- Horizontal axis: $kT/\varepsilon_F$
- Vertical axis: $\mu/\varepsilon_F$
- Point on the curve: $kT/\varepsilon_F = 1$, $\mu/\varepsilon_F = 1$
Sommerfeld Expansion - cnt.

\[ N = \int_{0}^{\infty} g(\epsilon) \tilde{n}_{FD}(\epsilon) \, d\epsilon \quad g(\epsilon) = \frac{3n}{2\epsilon_{F}^{3/2}} \sqrt{\epsilon} = g_{0} \sqrt{\epsilon} \quad (20) \]

\[ N = g_{0} \int_{0}^{\infty} \sqrt{\epsilon} \tilde{n}_{FD}(\epsilon) \, d\epsilon \quad (21) \]

\[ N = g_{0} \int_{0}^{\infty} \tilde{n}_{FD}(\epsilon) \, d(\epsilon^{3/2}) \cdot \frac{2}{3} \quad (22) \]

\[ N = \frac{2}{3} g_{0} \left[ \tilde{n}_{FD}(\epsilon) \epsilon^{3/2} \bigg|_{\infty}^{0} - \int_{0}^{\infty} \epsilon^{3/2} \, d\tilde{n}_{FD}(\epsilon) \right] \quad (23) \]

\[ \frac{d\tilde{n}_{FD}(\epsilon)}{d\epsilon} = \frac{1}{kT} \frac{d}{d[\frac{(\epsilon - \mu)}{kT}]} \frac{1}{e^{(\epsilon - \mu)/kT} + 1} \quad (24) \]

\[ \frac{d\tilde{n}_{FD}(\epsilon)}{d\epsilon} = \frac{1}{kT} \frac{d}{dx} \frac{1}{e^{x} + 1} = \frac{1}{kT} \frac{-e^{x}}{(e^{x} + 1)^{2}} \quad (25) \]
Sommerfeld Expansion - cnt.

\[-\frac{e^x}{(e^x+1)^2}\] looks like the dashed curve. It is the derivative of the solid curve.
Sommerfeld Expansion - cnt.

So,

\[ N = \frac{2}{3} g_0 \int_{0}^{\infty} \epsilon^{3/2} \frac{1}{kT} \frac{e^x}{(e^x + 1)^2} d\epsilon \]  \hspace{1cm} (26)

Using \( x = \frac{\epsilon - \mu}{kT} \)

\[ N = \frac{2}{3} g_0 \int_{-\mu/kT}^{\infty} \epsilon^{3/2} \frac{e^x}{(e^x + 1)^2} dx \]  \hspace{1cm} (27)

Now we need to look at the approximation which we can made for \( \epsilon \) at \( \epsilon \approx \mu \).

\[ \epsilon^{3/2} = \mu^{3/2} + (\epsilon - \mu) \frac{d}{d\epsilon} \epsilon^{3/2} \bigg|_{\epsilon=\mu} + \frac{1}{2} (\epsilon - \mu)^2 \frac{d^2}{d\epsilon^2} \epsilon^{3/2} \bigg|_{\epsilon=\mu} + \cdots \]  \hspace{1cm} (28)

\[ \epsilon^{3/2} = \mu^{3/2} + \frac{3}{2} (\epsilon - \mu) \mu^{1/2} + \frac{3}{8} (\epsilon - \mu)^2 \mu^{-1/2} + \cdots \]  \hspace{1cm} (29)

So, the integration becomes:

\[ N = \frac{2}{3} g_0 \int_{-\mu/kT}^{\infty} \frac{e^x}{(e^x + 1)^2} \left[ \mu^{3/2} + \frac{3}{2} (\epsilon - \mu) \mu^{1/2} + \frac{3}{8} (\epsilon - \mu)^2 \mu^{-1/2} + \cdots \right] dx \]
Sommerfeld Expansion - cnt.

\[ N = \frac{2}{3} g_0 \int_{-\mu/kT}^{\infty} \frac{e^x}{(e^x+1)^2} \left[ \mu^{3/2} + \frac{3}{2} (xkT) \mu^{1/2} + \frac{3}{8} (xkT)^2 \mu^{-1/2} + \cdots \right] dx \]

Change the limits of the integration:

\[ N \approx \frac{2}{3} g_0 \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} \left[ \mu^{3/2} + \frac{3}{2} (xkT) \mu^{1/2} + \frac{3}{8} (xkT)^2 \mu^{-1/2} + \cdots \right] dx \]

This is valid - See the figure of \( \frac{d\bar{n}_{FD}}{d\epsilon} \) below: \( kT = 1.0/40.0 \text{ eV}, \mu = 6.0 \text{ eV} \).
Sommerfeld Expansion - cnt.

\[ \mathcal{N} \approx \frac{2}{3} g_0 \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} \left[ \mu^{3/2} + \frac{3}{2} (xkT)\mu^{1/2} + \frac{3}{8} (xkT)^2 \mu^{-1/2} + \cdots \right] \, dx \]

\[ \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} \, dx = \int_{-\infty}^{\infty} -\frac{d\bar{n}_{FD}}{d\epsilon} \, d\epsilon \\
= \bar{n}_{FD}(-\infty) - \bar{n}_{FD}(\infty) = 1 - 0 = 1 \quad (30) \]

\[ \int_{-\infty}^{\infty} \frac{xe^x}{(e^x+1)^2} \, dx = \int_{-\infty}^{\infty} \frac{x}{(e^x + 1)(1 + e^{-x})} \, dx = 0 \quad (31) \]

\[ \int_{-\infty}^{\infty} \frac{x^2e^x}{(e^x+1)^2} \, dx = \frac{\pi^2}{3} \quad (32) \]

\[ \int_{-\infty}^{\infty} \frac{x^3e^x}{(e^x+1)^2} \, dx \]

Therefore, we have [using \( g_0 = 3N/(2\epsilon_F^{3/2}) \)]:

\[ \mathcal{N} \approx \frac{2}{3} g_0 \mu^{3/2} + \frac{1}{4} g_0 (kT)^2 \mu^{-1/2} \frac{p^2}{3} + \cdots \]

\[ \mathcal{N} \approx \mathcal{N} \left( \frac{\mu}{\epsilon_F} \right)^{3/2} + \mathcal{N} \frac{\pi^2}{8} \frac{(kT)^2}{\epsilon_F^{3/2} \mu^{1/2}} + \cdots \quad (34) \]
Sommerfeld Expansion - cnt.

\[ 1 \approx \left( \frac{\mu}{\epsilon_F} \right)^{3/2} + \frac{\pi^2}{8} \frac{(kT)^2}{\epsilon_F^{3/2} \mu^{1/2}} + \cdots \]  \hfill (35)

\[ \left( \frac{\mu}{\epsilon_F} \right)^{3/2} \approx 1 - \frac{\pi^2}{8} \frac{(kT)^2}{\epsilon_F^{3/2} \mu^{1/2}} + \cdots \]  \hfill (36)

\[ \left( \frac{\mu}{\epsilon_F} \right)^{3/2} \approx 1 - \frac{\pi^2}{8} \frac{(kT)^2}{\epsilon_F^2} + \cdots \]  \hfill (37)

\[ \frac{\mu}{\epsilon_F} \approx \left[ 1 - \frac{\pi^2}{8} \frac{(kT)^2}{\epsilon_F^2} + \cdots \right]^{2/3} \]  \hfill (38)

\[ \frac{\mu}{\epsilon_F} \approx 1 - \frac{2}{3} \frac{\pi^2}{8} \left( \frac{kT}{\epsilon_F} \right)^2 + \cdots \]  \hfill (39)

\[ \frac{\mu}{\epsilon_F} \approx 1 - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 + \cdots \]  \hfill (40)
\[ \mu \approx \epsilon_F \] is because the second term \( \frac{\pi^2}{8} \frac{1}{\epsilon_F^{3/2} \mu^{1/2}} \) is a rather small correction.

Following the same steps, one can get the total energy:

\[ U \approx \frac{3}{5} N \mu^{5/2} \epsilon_F^{3/2} + \frac{3\pi^2}{8} N \frac{(kT)^2}{\epsilon_F} + \cdots \quad (41) \]

Using Eq. (40),

\[ U \approx \frac{3}{5} N \epsilon_F + \frac{\pi^2}{4} N \frac{(kT)^2}{\epsilon_F} + \cdots \quad (42) \]
Heat Capacity of Electron Gas

We can calculate the heat capacity of electron gas:

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V$$

Using Eq. (42),

$$C_V = \frac{\pi^2 N k^2 T}{2\epsilon_F}$$  \hspace{1cm} (44)$$

which is Eq. (7.48) on page 278 in the textbook.
Exercise 07-05: Sommerfeld Expansion (Problem 7.29):
Carry out the Sommerfeld Expansion for the energy integral to obtain Eq.(45). Then, plug in the expansion for $\mu$, Eq.(40) to obtain Eq.(46).

\[
U \approx \frac{3}{5} N \frac{\mu}{\epsilon_F^{3/2}} + \frac{3\pi^2}{8} N \frac{(kT)^2}{\epsilon_F} + \cdots \tag{45}
\]

\[
U \approx \frac{3}{5} N \epsilon_F + \frac{\pi^2}{4} N \frac{(kT)^2}{\epsilon_F} + \cdots \tag{46}
\]