1 Introductory Remarks

- Introduce myself
- Have students introduce themselves to learn names, year in grad school, and expectations
- The purpose of this course is to deepen your understanding of the theory of electromagnetism and (perhaps more importantly) to equip and familiarize you with a set of mathematical tools that will be valuable in many subfields of physics. It will be difficult, but I believe you all have the ability to be successful.
- We will not cover the entire textbook or often all topics in a chapter.
- Structure: Each class period divided in 2 with a 10 minute break. One half of each period (usually the second half of Monday) will be more of a “recitation” with time for questions about the book, homework, etc.
- Homework will be given out each week on Wednesday and due the following Wednesday.
- Working together on homework is allowed and strongly encouraged. Most of the learning in this class will probably take place while solving the homework problems.
- SDSU and USD students will give their assignments to their dept. secretary, who will mail it here. I will grade and return them the same way.

2 Spherical and Cylindrical Coordinate Conventions

We will use the angle symbols and conventions given in Jackson for spherical (p. 96) and cylindrical (p. 112) coordinates

In cylindrical coordinates, the infinitesimal displacement $d\vec{l} = (dr, r d\phi, dz)$

$$d^3 \vec{x} = r dr d\phi dz$$
Thus, an integral over all space is
\[ \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \int_{z=-\infty}^{\infty} f(r, \phi, z) r dr d\phi dz \]

In spherical coordinates, \( d\vec{l} = (dr, r d\theta, r \sin \theta d\phi) \)
\[ d^3 \vec{x} = r^2 \sin \theta dr d\theta d\phi \]

Thus, an integral over all space is
\[ \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi \]

3 Dirac Delta Function

Useful for incorporating point charges into continuous distributions. This “function” is only intended for use under the integral sign.
\[ \delta(x - a) = 0 \text{ for } x \neq a \]
\[ \int \delta(x - a) dx = 1 \text{ if the region of integration includes } a, \text{ and is 0 otherwise} \]
Most importantly, \( \int f(x) \delta(x - a) dx = f(a) \text{ if the region of integration includes } a \)

We can generalize this to three dimensions, so
\[ \int_{\Delta V} \delta(\vec{x} - \vec{X}) d^3 x = \begin{cases} 1 & \text{if } \Delta V \text{ contains } \vec{x} = \vec{X} \\ 0 & \text{otherwise} \end{cases} \]

4 Coulomb’s Law and Electric Field in Vector Form

We probably all remember Coulomb’s law as
\[ F = \frac{1}{4\pi \epsilon_0} \frac{q_1 q_2}{r^2} \]

However, this is as scalar force, and we will be working with the full vector calculus incarnation of electrodynamics, which means we need to make the equation look a little strange give it directionality.
\[ \vec{F} = \frac{q_1 q_2}{4\pi \epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \]

As a reminder, for real vectors \( |\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} \)
In cartesian coordinates, this means
\[ |\vec{x}|^3 = (x^2 + y^2 + z^2)^{\frac{3}{2}} \]

We also define the electric field such that \( \vec{F} = q \vec{E} \), where \( q \) is an infinitesimal test charge. This is physically unrealizable, but it helps with the math greatly.
This means
\[ \vec{E}(\vec{x}) = \frac{q_1}{4\pi \epsilon_0} \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} \]

The linearity of electric fields means that the field at point \(\vec{x}\) from a set of charges is the vector sum of the fields from the individual charges. Mathematically, this means
\[ \vec{E}(\vec{x}) = \sum_{i=1}^{n} \frac{q_i}{4\pi \epsilon_0} \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3} \]

We can further generalize to a continuous charge distribution \(\rho(\vec{x}')\)
\[ \vec{E}(\vec{x}) = \frac{1}{4\pi \epsilon_0} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x' \]

Let’s now use this as a quick example of the usefulness of the Dirac delta. A point charge \(q\) at the point \(\vec{x}_1\) can be represented as \(\rho(\vec{x}') = q\delta(\vec{x}' - \vec{x}_1)\)

The electric field is thus
\[ \vec{E}(\vec{x}) = \frac{1}{4\pi \epsilon_0} \int q\delta(\vec{x}' - \vec{x}_1) \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x' = \frac{q}{4\pi \epsilon_0} \int \delta(\vec{x}' - \vec{x}_1) \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x' \]
\[ = \vec{E}(\vec{x}) = \frac{q}{4\pi \epsilon_0} \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} \]

5 Gauss’s Law

Who remembers Gauss’s Law?

We start with an arbitrary closed surface surrounding a point charge (draw).

We want to find the flux of the electric field though an infinitesimal area \(d\vec{a}\), where \(d\vec{a} = \hat{n} da\)

\[ \vec{E} \cdot \hat{n} da = |\vec{E}||\hat{n}| \cos \theta da \]

By definition \(\hat{n} \equiv 1\).

\[ |\vec{E}| = \frac{q}{4\pi \epsilon_0} \frac{|\vec{x} - \vec{x}_1|}{|\vec{x} - \vec{x}_1|^3} = \frac{q}{4\pi \epsilon_0} \frac{1}{r^2} \]

where we have defined \(r \equiv |\vec{x} - \vec{x}_1|\).

\[ \vec{E} \cdot \hat{n} da = \frac{q}{4\pi \epsilon_0} \frac{\cos \theta}{r^2} da \]

We now need the definition of solid angle, which is \(4\pi\) times the fraction of the area of the sphere of radius \(r\) subtended by \(da\). To “tilt” \(da\) into the fictitious sphere, we use \(\cos \theta da\). The area of a sphere at \(r\) is \(4\pi\), so

\[ d\Omega = 4\pi \frac{da \cos \theta}{4\pi r^2} \Rightarrow da \cos \theta = r^2 d\Omega \]
Substituting this into the formula above yields

\[ \vec{E} \cdot \hat{n} \, da = \frac{q}{4\pi\epsilon_0} \, d\Omega \]

We can now integrate over the closed surface easily

\[ \oint \vec{E} \cdot \hat{n} \, da = \frac{q}{4\pi\epsilon_0} \oint d\Omega = \frac{q}{\epsilon_0} \]

Again, the linearity of electric fields allows us to generalize to a set of point charges inside a surface

\[ \oint \vec{E} \cdot \hat{n} \, da = \frac{1}{\epsilon_0} \sum_i q_i \]

and a continuous charge distribution

\[ \oint \vec{E} \cdot \hat{n} \, da = \frac{1}{\epsilon_0} \int_V \rho(x) \, d^3x \]

**Example 2.2 from Griffiths (p. 70)**

What is the field around a uniformly charged sphere of radius \( R \) with charge \( q \)? Draw sphere.

To make use of Gauss’s law, we need to draw a Gaussian surface, which can be any surface we want. We want to choose a surface such that \( \vec{E} \cdot \hat{n} \) becomes as simple as possible. We exploit the spherical symmetry of this problem to do so. The electric field will point radially outward and will thus have the form

\[ \vec{E} = |\vec{E}| \hat{r} \]

We also know that since the volume contains the entire charge distribution, the RHS of Gauss’s law will simply be \( q/\epsilon_0 \). Therefore, we have

\[ \oint_s |\vec{E}| \hat{r} \cdot \hat{n} \, da = \]

### 6 \( \nabla \)

From vector calculus, we have the operator “del” \( \nabla \) in \LaTeX

In cartesian coordinates

\[ \nabla \equiv \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \]

The definitions in cylindrical and spherical coordinates are on the back cover of Jackson. Applying this to a vector field gives us the divergence

\[ \nabla \cdot \vec{A} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (A_x, A_y, A_z) = \frac{\partial A_x}{\partial x}, \frac{\partial A_y}{\partial y}, \frac{\partial A_z}{\partial z} \]
which is a scalar quantity. Applying it to a function $\psi(x, y, z)$ is called the gradient leads to a vector field

$$\nabla \psi(x, y, z) = \frac{\partial \psi}{\partial x} \hat{x} + \frac{\partial \psi}{\partial y} \hat{y} + \frac{\partial \psi}{\partial z} \hat{z}$$

### 7 Differential form of Gauss’s Law

To obtain the differential form of Gauss’s law, we recall the divergence theorem from vector calculus

$$\oint_S \vec{A} \cdot \hat{n} \, da = \int_V \nabla \cdot \vec{A} \, d^3x$$

Simple substitution leads to

$$\oint \vec{E} \cdot \hat{n} \, da = \int_V \nabla \cdot \vec{E} \, d^3x = \int_V \frac{\rho(\vec{x})}{\epsilon_0} \, d^3x$$

Since this holds for any arbitrary volume:

$$\nabla \cdot \vec{E} = \frac{\rho(\vec{x})}{\epsilon_0}$$

### 8 Scalar Potential and $\nabla \times \vec{E}$

We now want to find the curl of an electric field, which is the cross product of $\nabla \times \vec{E}$.

We know that Stoke’s theorem says

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot \hat{n} \, da$$

So, if we can find

$$\oint_C \vec{E} \cdot d\vec{l}$$

we will be able to calculate $\nabla \times \vec{E}$.

Recall Coulomb’s law

$$\vec{E}(\vec{x}) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

We can figure out the curl and, as a bonus, define scalar potential $\Phi$ by taking a particular gradient

$$\nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right)$$

$$|\vec{x} - \vec{x}'| = \sqrt{(\vec{x} - \vec{x}') \cdot (\vec{x} - \vec{x}')} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$
\[
\frac{\partial}{\partial x} \frac{1}{|\vec{x} - \vec{x}'|} = \frac{\partial}{\partial x} \left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{1}{2}} = -\frac{1}{2} \left[ \cdots \right]^{-\frac{3}{2}} \cdot 2(x-x')
\]

Similarly for \(y\) and \(z\).

Notice that

\[
\left[ \cdots \right]^{-\frac{3}{2}} = \frac{1}{|\vec{x} - \vec{x}'|^3}
\]

Therefore,

\[
\nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = \frac{(-1)(x-x') \cdot \hat{x}}{|\vec{x} - \vec{x}'|^3} + \frac{(-1)(y-y') \cdot \hat{y}}{|\vec{x} - \vec{x}'|^3} + \frac{(-1)(z-z') \cdot \hat{z}}{|\vec{x} - \vec{x}'|^3}
\]

\[
\Rightarrow -\nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}
\]

Going back to Coulomb’s Law,

\[
\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x' = \vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \left( -\nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \right) d^3x'
\]

Since \(\nabla\) does not contain \(x'\), we can pull it out of the integral to yield

\[
\vec{E}(\vec{x}) = \frac{-1}{4\pi\epsilon_0} \nabla \int \rho(\vec{x}') \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'
\]

Notice that the result of the integral is a scalar, not a vector. So, we can define the electric field in terms of the scalar potential as \(\vec{E} = -\nabla \Phi\), where

\[
\Phi = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'
\]

9 Potential Difference \(V\)

From freshman physics and the first lecture we know that \(\vec{F} = q\vec{E}\). To determine the potential difference between two points in an electric field, we also need the definition of work \(W = Fd\), or in our vector world

\[
W = -\int_A^B \vec{F} \cdot d\vec{l} = -q \int_A^B \vec{E} \cdot d\vec{l} = \int_A^B \nabla \Phi \cdot d\vec{l} = q(\Phi_B - \Phi_A)
\]

We can thus define a potential difference \(V\) as

\[
V = -(\Phi_B - \Phi_A) = \int_A^B \vec{E} \cdot d\vec{l}
\]

If time permits, Example 2.6 from Griffiths (p. 85-86)